

# Invariant Algebras

Keqin Liu

Department of Mathematics  
The University of British Columbia  
Vancouver, BC  
Canada, V6T 1Z2

January, 2011

## Abstract

We introduce invariant algebras and representation<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> of algebras, and give many ways of constructing Lie algebras, Jordan algebras, Leibniz algebras, pre-Lie algebras and left-symmetric algebras in an invariant algebras.

In this paper, we introduce invariant algebras and representation<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> of algebras. The main property of invariant algebras is that an invariant algebra carries 14 associative products, which can be used to construct many different algebras inside an invariant algebra. We use this paper to give 6 ways of constructing a Lie algebra structure, 6 ways of constructing a Jordan algebra structure, 4 ways of constructing a Leibniz algebra structure, 14 ways of constructing a pre-Lie algebra structure and 14 ways of constructing a left-symmetric algebra structure inside an invariant algebra. Based on the main property of invariant algebras, we introduce representation<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> of algebras in the last section of this paper. Using representation<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> of algebras is a natural way of developing representation theory for some non-associative algebras like Leibniz algebras, pre-Lie algebras and left-symmetric algebras.

Throughout, all vector spaces are vector spaces over a field  $\mathbf{k}$ .

## 1 Basic Definitions

The notion of invariant algebras introduced in this paper is given in the following

**Definition 1.1** *Let  $A$  be an associative algebra with an idempotent  $q$ . The (right) invariant algebra  $(A, q)$  induced by the idempotent  $q$  is defined by*

$$(A, q) := \{x, \mid x \in A \text{ and } qxq = qx\}. \quad (1)$$

The most important example of invariant algebras is the linear invariant algebra over a vector space. Let  $V$  be a vector space, let  $End(V)$  be the associative algebra of all linear transformations from  $V$  to  $V$ , and let  $I$  (or  $I_V$ ) be the identity linear transformation of  $V$ . If  $W$  is a subspace of  $V$ , then a linear transformation  $q$  satisfying

$$q(W) = 0 \quad \text{and} \quad (q - I)(V) \subseteq W \quad (2)$$

is an idempotent because

$$q^2(v) - q(v) = q(q - I)(v) \in q(W) = 0 \quad \text{for all } v \in V.$$

An element  $q$  of  $End(V)$  satisfying (2) is called a  **$W$ -idempotent**.

Note that

$$(End(V), q) = \{ f \mid f \in End(V) \text{ and } f(W) \subseteq W \}, \quad (3)$$

where  $W$  is a subspace of a vector space  $V$  and  $q$  is a  $W$ -idempotent. The invariant algebra  $(End(V), q)$  is called the **linear invariant algebra over  $V$**  induced by the  $W$ -idempotent  $q$ , which consists of all linear transformations of  $V$  having  $W$  as an invariant subspace by (3).

Let  $(A, q)$  be an invariant algebra. The set

$$(A, q)^{ann} := \{ qx - x \mid x \in (A, q) \} \quad (4)$$

is clearly a subspace of  $(A, q)$ . For  $a, x \in (A, q)$ , we have

$$(qx - x)a = q(xa) - (xa) \in (A, q)^{ann}$$

and

$$a(qx - x) = q(-aqx + ax) - (-aqx + ax) \in (A, q)^{ann}.$$

Hence, the subspace  $(A, q)^{ann}$  is an ideal of  $(A, q)$ , which is called the **annihilator** of  $(A, q)$ . For  $x \in (A, q)$ , we have

$$xq - x = q(-xq + x) - (-xq + x) \in (A, q)^{ann},$$

which implies that

$$(A, q)^{ann} \supseteq \{ xq - x \mid x \in (A, q) \} \cup \{ qx - xq \mid x \in (A, q) \}. \quad (5)$$

**Definition 1.2** Let  $(A, q_A)$  and  $(B, q_B)$  be invariant algebras. A map  $\phi : (A, q_A) \rightarrow (B, q_B)$  is called an **invariant homomorphism** if

$$\phi(x + y) = \phi(x) + \phi(y), \quad \phi(xy) = \phi(x)\phi(y) \quad \text{for } x, y \in (A, q_A)$$

and

$$\phi(1_A) = 1_B, \quad \phi(q_A) = q_B,$$

where  $1_A$  and  $1_B$  are the identities of  $A$  and  $B$ , respectively. A bijective invariant homomorphism is called an **invariant isomorphism**.

The next proposition shows that any invariant algebra can be embedded in a linear invariant algebra.

**Proposition 1.1** *If  $(A, q)$  is an invariant algebra, then the map*

$$\phi : a \mapsto a_L \quad \text{for } a \in (A, q)$$

*is an injective invariant homomorphism from  $(A, q)$  to the linear invariant algebra  $(\text{End}(A, q), q_L)$  over  $(A, q)$  induced by the  $(A, q)^{\text{ann}}$ -idempotent  $q_L$ , where  $a_L$  is the left multiplication defined by*

$$a_L(x) := ax \quad \text{for } x \in (A, q).$$

## 2 Hu-Liu Products

We now define Hu-Liu products in the following

**Definition 2.1** *Let  $(A, q)$  be an invariant algebra over a field  $\mathbf{k}$ . For  $1 \leq i \leq 14$ , the  $i$ -th **Hu-Liu product**  $\circ_i$  is defined by*

$$x \circ_1 y : = kyx + hyqx, \tag{6}$$

$$x \circ_2 y : = kyx - kyqx + qxy, \tag{7}$$

$$x \circ_3 y : = yx + kyqx - yxq, \tag{8}$$

$$x \circ_4 y : = yx + kqxy - yxq, \tag{9}$$

$$x \circ_5 y : = kyx - kyqx + qyx, \tag{10}$$

$$x \circ_6 y : = yx + kqyx - yxq, \tag{11}$$

$$x \circ_7 y : = kyx + xqy - kyxq, \tag{12}$$

$$x \circ_8 y : = kxy + hxqy, \tag{13}$$

$$x \circ_9 y : = xy - xqy + kqxy, \tag{14}$$

$$x \circ_{10} y : = xy - xqy + kqyx, \tag{15}$$

$$x \circ_{11} y : = xy - xyq + kqxy, \tag{16}$$

$$x \circ_{12} y : = xy + kyqx - xyq, \tag{17}$$

$$x \circ_{13} y : = xy - xyq + kqyx, \tag{18}$$

$$x \circ_{14} y : = xy + kxqy - xyq, \tag{19}$$

where  $k, h$  are fixed scalars in the field  $\mathbf{k}$  and  $x, y \in (A, q)$ .

For  $i = 1$  or  $8$ ,  $\circ_i$  is also denoted by  $\circ_{i,k,h}$  to indicate that  $\circ_1$  and  $\circ_8$  depend on the scalars  $k$  and  $h$ . Similarly, for  $7 \geq i \geq 2$  or  $14 \geq i \geq 9$ ,  $\circ_i$  is also denoted by  $\circ_{i,k}$ .

The next proposition gives the main property of 14 Hu-Liu products.

**Proposition 2.1** *If  $(A, q)$  is an invariant algebra over a field  $\mathbf{k}$ , then the  $i$ -th Hu-Liu product  $\circ_i$  satisfies the associative law:*

$$(x \circ_i y) \circ_i z = x \circ_i (y \circ_i z), \quad (20)$$

where  $1 \leq i \leq 14$  and  $x, y, z \in (A, q)$ .

### 3 Lie Algebras

If  $(A, q)$  is an invariant algebra, then  $(A, q)$  can be made into a Lie algebra by the well-known square bracket  $[\cdot, \cdot]$ , where  $[\cdot, \cdot]$  is defined by

$$[x, y] := xy - yx \quad \text{for } x, y \in (A, q). \quad (21)$$

Except the ordinary square bracket (21), there are other six square brackets which also make an invariant algebra into a Lie algebra. We now introduce the six square brackets in the following

**Definition 3.1** *Let  $(A, q)$  be an invariant algebra over a field  $\mathbf{k}$ . For  $1 \leq i \leq 7$ , the  $i$ -th square bracket  $[\cdot, \cdot]_i$  is defined by*

$$[x, y]_1 : = qxy - qyx, \quad (22)$$

$$[x, y]_2 : = xqy - yqx, \quad (23)$$

$$[x, y]_3 : = xy - yx + kxqy - kyqx, \quad (24)$$

$$[x, y]_4 : = xy - yx - xqy + yqx - kqxy + kqyx, \quad (25)$$

$$[x, y]_5 : = xy - yx - xyq + yxq - kqxy + kqyx, \quad (26)$$

$$[x, y]_6 : = xy - yx - xyq + yxq + kxqy - kyqx, \quad (27)$$

where  $x, y \in (A, q)$ , and  $k$  is a fixed scalar in the field  $\mathbf{k}$ .

The square bracket  $[\cdot, \cdot]_i$  with  $3 \leq i \leq 6$  is also denoted by  $[\cdot, \cdot]_{i,k}$  to indicate its dependence on the scalar  $k$ . The next proposition gives the basic property of the six square brackets.

Each  $i$ -th square bracket with  $6 \geq i \geq 1$  can be expressed in terms of the Hu-Liu product  $\circ_i$  by a few ways. One of the ways is given by

$$\begin{aligned} [x, y]_1 &= x \circ_{2,0} y - y \circ_{2,0} x, & [x, y]_2 &= x \circ_{8,0,1} y - y \circ_{8,0,1} x, \\ [x, y]_{3,k} &= x \circ_{8,1,k} y - y \circ_{8,1,k} x, & [x, y]_{4,k} &= x \circ_{10,k} y - y \circ_{10,k} x, \\ [x, y]_{5,k} &= x \circ_{13,k} y - y \circ_{13,k} x, & [x, y]_{6,k} &= x \circ_{14,k} y - y \circ_{14,k} x. \end{aligned}$$

**Proposition 3.1** *Let  $(A, q)$  be an invariant algebra.*

(i) The 1-st and 2-nd square brackets satisfy the Jacobi identity; that is,

$$[[x, y]_i, z]_i + [[y, z]_i, x]_i + [[z, x]_i, y]_i = 0 \quad \text{for } x, y, z \in (A, q), \quad (28)$$

where  $i = 1$  and  $2$ .

(ii) The 3-rd square bracket satisfies the following **long Jacobi-like identity**<sup>1-st</sup>:

$$\begin{aligned} & [[x, y]_h, z]_k + [[y, z]_h, x]_k + [[z, x]_h, y]_k + \\ & + [[x, y]_k, z]_h + [[y, z]_k, x]_h + [[z, x]_k, y]_h = 0 \end{aligned} \quad (29)$$

for  $x, y, z \in (A, q)$  and  $h, k \in \mathbf{k}$ .

(iii) Let  $x, y, z \in (A, q)$  and  $k, h \in \mathbf{k}$ . If  $i = 4, 5$  and  $6$ , then the  $i$ -th angle bracket satisfies the **Jacobi-like identity**<sup>1-st</sup>:

$$[[x, y]_{i,h}, z]_{i,k} + [[y, z]_{i,h}, x]_{i,k} + [[z, x]_{i,h}, y]_{i,k} = 0. \quad (30)$$

Moreover, we have

$$[[x, y]_{i,h}, z]_{i,k} = [[x, y]_{i,k}, z]_{i,h} \quad \text{for } i = 4, 5 \text{ and } 6. \quad (31)$$

Since the six square brackets are anti-commutative, each of the six square brackets makes an invariant algebra into a Lie algebra.

## 4 Jordan Algebras

We begin this section by recalling the definition of a Jordan algebra from [2].

**Definition 4.1** A Jordan algebra  $J$  is an algebra over a field  $\mathbf{k}$  of characteristic  $\neq 2$  with a product composition  $\odot$  satisfying

$$x \odot y = y \odot x \quad (\text{Commutative Law}) \quad (32)$$

and

$$\left( (x \odot x) \odot y \right) \odot x = (x \odot x) \odot (y \odot x) \quad (\text{Jordan Identity}) \quad (33)$$

for all  $x, y \in J$ .

Based on the  $i$ -th square brackets in Definition 3.1, we have the following

**Proposition 4.1** An invariant algebra  $(A, q)$  over a field  $\mathbf{k}$  of characteristic  $\neq 2$  is a Jordan algebra under each of the following 6 products:

$$x \odot_1 y = \frac{1}{2}(qxy + qyx), \quad (34)$$

$$x \odot_2 y = \frac{1}{2}(xqy + yqx), \quad (35)$$

$$x \odot_3 y = \frac{1}{2}(xy + kxqy + yx + kyqx), \quad (36)$$

$$x \odot_4 y = \frac{1}{2}(xy - xqy - kqxy + yx - yqx - kqyx), \quad (37)$$

$$x \odot_5 y = \frac{1}{2}(xy - xyq - kqxy + yx - yxq - kqyx), \quad (38)$$

$$x \odot_6 y = \frac{1}{2}(xy - xyq + kxqy + yx - yxq + kyqx), \quad (39)$$

where  $x, y \in (A, q)$ , and  $k$  is a fixed scalar in the field  $\mathbf{k}$ .

## 5 Leibniz Algebras

We begin this section with the definition of  $i$ -th angle brackets.

**Definition 5.1** Let  $(A, q)$  be an invariant algebra over a field  $\mathbf{k}$ . For  $i = 1, 2, 3$  and 4, the  $i$ -th angle bracket  $\langle \cdot, \cdot \rangle_i$  is defined by

$$\langle x, y \rangle_1 = xqy - qyx, \quad (40)$$

$$\langle x, y \rangle_2 = xy - yx + yqx - xyq + kqyx - kqxy, \quad (41)$$

$$\langle x, y \rangle_3 = xy - yx - xyq + yxq + kxqy - kqyx, \quad (42)$$

$$\langle x, y \rangle_4 = xy - yx + yqx - xyq + kxqy - kqyx, \quad (43)$$

where  $x, y \in (A, q)$ , and  $k$  is a fixed scalar in the field  $\mathbf{k}$ .

The angle bracket  $\langle \cdot, \cdot \rangle_i$  with  $2 \leq i \leq 4$  is also denoted by  $\langle \cdot, \cdot \rangle_{i,k}$  to indicate its dependence on the scalar  $k$ . The next proposition gives the basic property of the four angle brackets.

**Proposition 5.1** Let  $(A, q)$  be an invariant algebra.

(i) The 1-st angle bracket  $\langle x, y \rangle_1$  satisfies the (right) Leibniz identity; that is,

$$\langle x, \langle y, z \rangle_1 \rangle_1 = \langle \langle x, y \rangle_1, z \rangle_1 - \langle \langle x, z \rangle_1, y \rangle_1 \quad (44)$$

for  $x, y, z \in (A, q)$ .

(ii) Let  $x, y, z \in (A, q)$  and  $k, h \in \mathbf{k}$ . If  $i = 2, 3$  and 4, then the  $i$ -th angle bracket satisfies the **Jacobi-like identity**<sup>2-nd</sup>:

$$\langle x, \langle y, z \rangle_{i,k} \rangle_{i,h} = \langle \langle x, y \rangle_{i,h}, z \rangle_{i,k} - \langle \langle x, z \rangle_{i,k}, y \rangle_{i,h}. \quad (45)$$

Moreover, we have

$$\langle x, \langle y, z \rangle_{i,k} \rangle_{i,h} = \langle x, \langle y, z \rangle_{i,h} \rangle_{i,k} \quad \text{for } i = 2, 3 \text{ and } 4 \quad (46)$$

and

$$\langle \langle x, y \rangle_{i,k}, z \rangle_{i,h} = \langle \langle x, y \rangle_{i,h}, z \rangle_{i,k} \quad \text{for } i = 2 \text{ and } 3. \quad (47)$$

Following [3], the notion of (right) Leibniz algebras is given in the following

**Definition 5.2** *A vector space  $L$  is called a **(right) Leibniz algebra** if there exists a binary operation  $\langle \cdot, \cdot \rangle: L \times L \rightarrow L$ , called the **angle bracket**, such that the **(right) Leibniz identity** holds:*

$$\langle \langle x, y \rangle, z \rangle = \langle x, \langle y, z \rangle \rangle + \langle \langle x, z \rangle, y \rangle \quad \text{for } x, y, z \in L. \quad (48)$$

By Proposition 5.1, each of the four angle brackets makes an invariant algebra into a (right) Leibniz algebra.

## 6 Pre-Lie Algebras

We begin this section by recalling the definition of a pre-Lie algebra from [1].

**Definition 6.1** *A **pre-Lie algebra**  $A$  is a vector space equipped with a **dot operation**  $\cdot : A \otimes A \rightarrow A$  which satisfy the following identity:*

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (x \cdot z) \cdot y - x \cdot (z \cdot y) \quad \text{for } x, y, z \in A. \quad (49)$$

We use  $(A, \cdot)$  to denote a pre-Lie algebra  $A$  equipped with a dot operation  $\cdot$ . Clearly, the square bracket

$$(x, y) \mapsto [x, y]^\cdot := x \cdot y - y \cdot x \quad \text{for } x, y \in A \quad (50)$$

satisfies the Jacobi identity; that is, a pre-Lie algebra is a Lie-admissible algebra. The square bracket  $[x, y]^\cdot$  defined by (50) is called the **accompanying square product**. The following proposition gives 14 ways of introducing a pre-Lie algebra structure in an invariant algebra and shows that each corresponding accompanying square product only differs from a square product in Definition 3.1 by a scalar.

**Proposition 6.1** *Let  $(A, q)$  be an invariant algebra over a field  $\mathbf{k}$ . Let  $k$  be a fixed scalar in the field  $\mathbf{k}$ .  $((A, q), \cdot_i)$  is a pre-Lie algebra if the dot operation  $\cdot_i$  is chosen in one of the following 14 ways:*

- (1)  $x \cdot_1 y = kyx + xqy - kyqx$ ,  $[x, y]^{\cdot_1} = [x, y]_2$  for  $k = 0$  and  $-\frac{1}{k}[x, y]^{\cdot_1} = [x, y]_3$  for  $k \neq 0$ ;
- (2)  $x \cdot_2 y = kyx + xqy - kyqx - qyx - qxy$ ,  $[x, y]^{\cdot_2} = [x, y]_2$  for  $k = 0$  and  $-\frac{1}{k}[x, y]^{\cdot_2} = [x, y]_3$  for  $k \neq 0$ ;
- (3)  $x \cdot_3 y = kyx + xqy + (1 - k)yqx - qyx$ ,  $[x, y]^{\cdot_3} = [x, y]_1$  for  $k = 0$  and  $-\frac{1}{k}[x, y]^{\cdot_3} = [x, y]_4$  for  $k \neq 0$ ;

- (4)  $x \cdot_4 y = kyx + xqy - qyx - qxy - kyxq$ ,  $[x, y]^{\cdot 4} = [x, y]_2$  for  $k = 0$  and  $-\frac{1}{k}[x, y]^{\cdot 4} = [x, y]_6$  for  $k \neq 0$ ;
- (5)  $x \cdot_5 y = kyx + xqy + yqx - qyx - kyxq$ ,  $-[x, y]^{\cdot 5} = [x, y]_1$  for  $k = 0$  and  $-\frac{1}{k}[x, y]^{\cdot 5} = [x, y]_5$  for  $k \neq 0$ ;
- (6)  $x \cdot_6 y = yx - xqy - yqx + xyq$  and  $-[x, y]^{\cdot 6} = [x, y]_5$ ;
- (7)  $x \cdot_7 y = yx + kxqy - yqx + xyq$  and  $-[x, y]^{\cdot 7} = [x, y]_6$ ;
- (8)  $x \cdot_8 y = yx + kxqy - yqx + xyq - (k+1)qyx - (k+1)qxy$  and  $-[x, y]^{\cdot 8} = [x, y]_6$ ;
- (9)  $x \cdot_9 y = xy + kyqx - xyq - yxq$  and  $[x, y]^{\cdot 9} = [x, y]_3$ ;
- (10)  $x \cdot_{10} y = xy + yqx - xyq + kqyx - yxq$  and  $[x, y]^{\cdot 10} = [x, y]_4$ ;
- (11)  $x \cdot_{11} y = xy + yqx - xyq + kqxy - yxq$  and  $[x, y]^{\cdot 11} = [x, y]_4$ ;
- (12)  $x \cdot_{12} y = xy + kxqy - xyq - kqyx - kqxy$  and  $[x, y]^{\cdot 12} = [x, y]_6$ ;
- (13)  $x \cdot_{13} y = xy + kxqy + kyqx - xyq - kqyx$  and  $[x, y]^{\cdot 13} = [x, y]_5$ ;
- (14)  $x \cdot_{14} y = xy + kxqy - (k+1)qyx - (k+1)qxy$  and  $[x, y]^{\cdot 14} = [x, y]_3$ ,
- where  $x, y \in (A, q)$ , and  $[x, y]_i$  is the  $i$ -th square product.

## 7 Left Symmetric Algebras

We begin this section by recalling the definition of a left symmetric algebra from [1].

**Definition 7.1** *A left symmetric algebra  $A$  is a vector space equipped with a dot operation  $\cdot : A \otimes A \rightarrow A$  which satisfy the following identity:*

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z \quad \text{for } x, y, z \in A. \quad (51)$$

We use  $(A, \cdot)$  to denote a left symmetric algebra  $A$  equipped with a dot operation  $\cdot$ . It is easy to check that the square bracket

$$(x, y) \mapsto [x, y]^{\cdot} := x \cdot y - y \cdot x \quad \text{for } x, y \in A \quad (52)$$

satisfies the Jacobi identity; that is, a left symmetric algebra is a Lie-admissible algebra. The square bracket  $[x, y]^{\cdot}$  defined by (52) is called the **accompanying square product**. The following proposition gives 14 ways of introducing a left symmetric algebra structure in an invariant algebra and shows that each corresponding accompanying square product only differs from a square product in Definition 3.1 by a scalar.



**Proposition 7.1** *Let  $(A, q)$  be an invariant algebra over a field  $\mathbf{k}$ . Let  $k$  be a fixed scalar in the field  $\mathbf{k}$ .  $((A, q), \cdot_i)$  is a left symmetric algebra if the dot operation  $\cdot_i$  is chosen in one of the following 14 ways:*

- (1)  $x \cdot_1 y = kxy - kxqy + yqx$ ,  $-[x, y]^{\cdot_1} = [x, y]_2$  for  $k = 0$  and  $\frac{1}{k}[x, y]^{\cdot_1} = [x, y]_3$  for  $k \neq 0$ ;
  - (2)  $x \cdot_2 y = kxy + (1 - k)xqy + yqx - qxy$ ,  $-[x, y]^{\cdot_2} = [x, y]_1$  for  $k = 0$  and  $\frac{1}{k}[x, y]^{\cdot_2} = [x, y]_4$  for  $k \neq 0$ ;
  - (3)  $x \cdot_3 y = xy + yqx - qxy$  and  $[x, y]^{\cdot_3} = [x, y]_4$ ;
  - (4)  $x \cdot_4 y = kxy - kxqy + yqx - qyx - qxy$ ,  $-[x, y]^{\cdot_4} = [x, y]_2$  for  $k = 0$  and  $\frac{1}{k}[x, y]^{\cdot_4} = [x, y]_3$  for  $k \neq 0$ ;
  - (5)  $x \cdot_5 y = kyx + (1 - k)yqx - qyx - qxy$ ,  $-[x, y]^{\cdot_5} = [x, y]_2$  for  $k = 0$  and  $\frac{1}{k}[x, y]^{\cdot_5} = [x, y]_3$  for  $k \neq 0$ ;
  - (6)  $x \cdot_6 y = kyx + yqx - qyx - qxy - kyxq$ ,  $-[x, y]^{\cdot_6} = [x, y]_2$  for  $k = 0$  and  $-\frac{1}{k}[x, y]^{\cdot_6} = [x, y]_6$  for  $k \neq 0$ ;
  - (7)  $x \cdot_7 y = kyx + xqy + yqx - qxy - kyxq$ ,  $-[x, y]^{\cdot_7} = [x, y]_1$  for  $k = 0$  and  $-\frac{1}{k}[x, y]^{\cdot_7} = [x, y]_5$  for  $k \neq 0$ ;
  - (8)  $x \cdot_8 y = kxy + yqx - kxyq - qyx - qxy$ ,  $-[x, y]^{\cdot_8} = [x, y]_2$  for  $k = 0$  and  $\frac{1}{k}[x, y]^{\cdot_8} = [x, y]_6$  for  $k \neq 0$ ;
  - (9)  $x \cdot_9 y = kxy + xqy + yqx - kxyq - qxy$ ,  $-[x, y]^{\cdot_9} = [x, y]_1$  for  $k = 0$  and  $\frac{1}{k}[x, y]^{\cdot_9} = [x, y]_5$  for  $k \neq 0$ ;
  - (10)  $x \cdot_{10} y = yx + xqy - xyq + kqxy - yxq$  and  $[x, y]^{\cdot_{10}} = [x, y]_4$ ;
  - (11)  $x \cdot_{11} y = yx + kxqy - xyq - yxq$  and  $-[x, y]^{\cdot_{11}} = [x, y]_3$ ;
  - (12)  $x \cdot_{12} y = yx + kxqy + (k - 1)yqx - xyq - (k - 1)qxy - yxq$  and  $-[x, y]^{\cdot_{12}} = [x, y]_4$ ;
  - (13)  $x \cdot_{13} y = yx + xqy - xyq + kqyx - yxq$  and  $-[x, y]^{\cdot_{13}} = [x, y]_4$ ;
  - (14)  $x \cdot_{14} y = yx + xqy + kyqx - xyq - kqyx - kqxy - yxq$  and  $-[x, y]^{\cdot_{14}} = [x, y]_3$ ,
- where  $x, y \in (A, q)$ , and  $[x, y]_i$  is the  $i$ -th square product.

## 8 Representation<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> of Algebras

Let  $(\mathcal{A}, \star)$  be an algebra (not necessarily associative algebra) over  $\mathbf{k}$  with a product  $\star$ . Based on invariant algebras, we introduce a representation<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> of an algebra  $(\mathcal{A}, \star)$  in the following

**Definition 8.1** Let  $W$  be a subspace of a vector space  $V$  over a field  $\mathbf{k}$  and let  $q$  be a  $W$ -idempotent. A linear map  $\varphi$  from an algebra  $(\mathcal{A}, \star)$  to the invariant algebra  $(\text{End}(V), q)$  is called a **representation**<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> of  $(\mathcal{A}, \star)$  induced by  $(q, W)$  if there exist the scalars  $c_1, \dots, c_8 \in \mathbf{k}$  such that

$$\begin{aligned} \varphi(x \star y) = & c_1 \varphi(x) \varphi(y) + c_2 \varphi(y) \varphi(x) + c_3 q \varphi(x) \varphi(y) + c_4 q \varphi(y) \varphi(x) + \\ & + c_5 \varphi(x) q \varphi(y) + c_6 \varphi(y) q \varphi(x) + c_7 \varphi(x) \varphi(y) q + c_8 \varphi(y) \varphi(x) q \end{aligned} \quad (53)$$

for all  $x, y \in \mathcal{A}$ .

The language of modules<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> is more convenient to state some facts about representations<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup>. We now introduce a module<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> over an algebra in the following

**Definition 8.2** Let  $W$  be a subspace of a vector space  $V$  over a field  $\mathbf{k}$  and let  $q$  be a  $W$ -idempotent.  $V$  is called a **module**<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> over an algebra  $(\mathcal{A}, \star)$  or a  **$\mathcal{A}$ -module**<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> induced by  $(q, W)$  if there is a map:  $(x, v) \mapsto x \cdot v$  from  $\mathcal{A} \times V$  to  $V$  such that

$$(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v), \quad (54)$$

$$x \cdot (av + bu) = a(x \cdot v) + b(x \cdot u), \quad (55)$$

$$\begin{aligned} (x \star y) \cdot v = & c_1 x \cdot y \cdot v + c_2 y \cdot x \cdot v + c_3 q \cdot x \cdot y \cdot v + c_4 q \cdot y \cdot x \cdot v + \\ & + c_5 x \cdot q \cdot y \cdot v + c_6 y \cdot q \cdot x \cdot v + c_7 x \cdot y \cdot q \cdot v + c_8 y \cdot x \cdot q \cdot v, \end{aligned} \quad (56)$$

$$x \cdot W := \{ x \cdot w \mid w \in W \} \subseteq W, \quad (57)$$

where  $x, y \in \mathcal{A}$ ,  $v, u \in V$ ,  $a, b \in \mathbf{k}$  and  $q \cdot v := q(v)$ .

A module<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> over an algebra  $(\mathcal{A}, \star)$  induced by  $(q, W)$  is also denoted by  $V_{(q, W)}$ . A subspace  $U$  of an  $\mathcal{A}$ -module<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup>  $V = V_{(q, W)}$  is called a **submodule**<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> of  $V$  if

$$x \cdot u \in U \quad \text{for } x \in \mathcal{A} \text{ and } u \in U \quad (58)$$

and

$$q \cdot u - u \in U \quad \text{for } u \in U. \quad (59)$$

Every  $\mathcal{A}$ -module<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup>  $V = V_{(q, W)}$  has at least three submodules<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup>:  $0$ ,  $W$  and  $V$ . An  $\mathcal{A}$ -module<sup>(c<sub>1</sub>,...,c<sub>8</sub>)</sup> with  $c_3 = \dots = c_8 = 0$  is called an  **$\mathcal{A}$ -module**<sup>(c<sub>1</sub>, c<sub>2</sub>)</sup>.

**Proposition 8.1** *If  $V_{(q,W)}$  is module $^{(c_1, \dots, c_8)}$  over an algebra  $(\mathcal{A}, \star)$ , then the subspace  $W$  is an  $\mathcal{A}$ -module $^{(c_2)}$  by restriction and the quotient space  $\frac{V}{W}$  is a  $\mathcal{A}$ -module $^{(c_1+c_3+c_5+c_7)}$  under the following action:*

$$x \cdot (v + W) := x \cdot v + W \quad \text{for } x \in \mathcal{A} \text{ and } v \in V. \quad (60)$$

We now define the building blocks for modules $^{(c_1, \dots, c_8)}$  over an algebra.

**Definition 8.3** *Let  $V_{(q,W)}$  be a module $^{(c_1, \dots, c_8)}$  over an algebra  $(\mathcal{A}, \star)$ .*

- (i)  *$V_{(q,W)}$  is said to be **2-irreducible** if  $V_{(q,W)} \neq 0$  and  $V_{(q,W)}$  has no submodules $^{(c_1, \dots, c_8)}$  which are not equal to 0 and  $V_{(q,W)}$ .*
- (ii)  *$V_{(q,W)}$  is said to be **3-irreducible** if  $W \neq 0$ ,  $V_{(q,W)} \neq W$  and  $V_{(q,W)}$  has no submodules $^{(c_1, \dots, c_8)}$  which are not equal to 0,  $W$  and  $V_{(q,W)}$ .*

The next proposition gives the basic properties of the building blocks.

**Proposition 8.2** *Let  $V = V_{(q,W)}$  be a module $^{(c_1, \dots, c_8)}$  over an algebra  $(\mathcal{A}, \star)$ .*

- (i) *If  $V_{(q,W)}$  is 2-irreducible, then either  $V = W$  in which case  $q = 0$  on  $V$  and  $V$  is a  $\mathcal{A}$ -module $^{(c_2)}$ , or  $W = 0$  in which case  $q = 1$  on  $V$  and  $V$  is a  $\mathcal{A}$ -module $^{(c_1+c_3+c_5+c_7)}$ .*
- (ii) *If  $V_{(q,W)}$  is 3-irreducible, then both the  $\mathcal{A}$ -module $^{(c_1)}$   $W$  and the  $\mathcal{A}$ -module $^{(c_1+c_3+c_5+c_7)}$   $\frac{V}{W}$  are 2-irreducible.*

**Definition 8.4** *Let  $(V_1)_{(q_1, W_1)}$  and  $(V_2)_{(q_2, W_2)}$  be two modules $^{(c_1, \dots, c_8)}$  over an algebra  $(\mathcal{A}, \star)$ .*

- (i) *A linear map  $\varphi : V_1 \rightarrow V_2$  is called a **homomorphism** $^{(c_1, \dots, c_8)}$  if*

$$\varphi(x \cdot v_1) = x \cdot \varphi(v_1) \quad \text{for } x \in L \text{ and } v_1 \in V_1 \quad (61)$$

*and*

$$\varphi(q_1 \cdot v_1) = q_2 \cdot \varphi(v_1) \quad \text{for } v_1 \in V_1. \quad (62)$$

- (ii) *If a homomorphism $^{(c_1, \dots, c_8)}$   $\varphi : V_1 \rightarrow V_2$  is bijective, then we say that  $\varphi$  is an **isomorphism** $^{(c_1, \dots, c_8)}$  and  $V_1$  is **isomorphic** to  $V_2$ .*

Let  $V = V_{(q,W)}$  be a module $^{(c_1, \dots, c_8)}$  over an algebra  $(\mathcal{A}, \star)$  induced by  $(q, W)$ . If  $U$  is a submodule $^{(c_1, \dots, c_8)}$  of  $V$ , then  $q|U$  is a  $(W \cap U)$ -idempotent and  $\bar{q} : \frac{V}{U} \rightarrow \frac{V}{U}$  is a  $\frac{U+W}{U}$ -idempotent, where  $\bar{q}$  is defined by

$$\bar{q} \cdot (v + U) = \bar{q} \cdot v + U \quad \text{for } v \in V. \quad (63)$$

Hence,  $U_{q|U, W \cap U}$  is an  $\mathcal{A}$ -module $^{(c_1, \dots, c_8)}$  induced by  $(q|U, W \cap U)$ , and the quotient space  $\frac{V}{U}$  is also a  $\mathcal{A}$ -module $^{(c_1, \dots, c_8)}$  under the module $^{(c_1, \dots, c_8)}$  action consisting of (63) and the following

$$x \cdot (v + U) := x \cdot v + U \quad \text{for } x \in \mathbf{A} \text{ and } v \in V. \quad (64)$$

The  $\mathcal{A}$ -module $^{(c_1, \dots, c_8)} \left( \frac{V}{U} \right)_{(\bar{q}, \frac{U+W}{U})}$  is called the **quotient  $\mathcal{A}$ -module** $^{(c_1, \dots, c_8)}$  with respect to submodule $^{(c_1, \dots, c_8)} U$ .

**Proposition 8.3** *Let  $(V_1)_{(q_1, W_1)}$  and  $(V_2)_{(q_2, W_2)}$  be two modules $^{(c_1, \dots, c_8)}$  over an algebra  $(\mathcal{A}, \star)$ , and suppose that  $\varphi : V_1 \rightarrow V_2$  is a homomorphism $^{(c_1, \dots, c_8)}$ .*

- (i)  $\varphi(W_1) \subseteq W_2$ .
- (ii) The **kernel**  $\text{Ker}\varphi := \{v_1 \in V_1 \mid \varphi(v_1) = 0\}$  is a submodule $^{(c_1, \dots, c_8)}$  of  $\mathcal{A}$ -module $^{(c_1, \dots, c_8)} V_1$ .
- (iii) The **image**  $\text{Im}\varphi := \{\varphi(v_1) \mid v_1 \in V_1\}$  is a submodule $^{(c_1, \dots, c_8)}$  of  $\mathcal{A}$ -module $^{(c_1, \dots, c_8)} V_2$ .
- (iv) The map  $\bar{\varphi} : \frac{V_1}{\text{Ker}\varphi} \rightarrow \text{Im}\varphi$  defined by

$$\varphi(v_1 + \text{Ker}\varphi) := \varphi(v_1) \quad \text{for } v_1 \in V_1 \quad (65)$$

is an isomorphism $^{(c_1, \dots, c_8)}$  from the quotient  $\mathcal{A}$ -module $^{(c_1, \dots, c_8)} \left( \frac{V_1}{\text{Ker}\varphi} \right)$  induced by  $\left( \bar{q}, \frac{\text{Ker}\varphi + W_1}{\text{Ker}\varphi} \right)$  to the submodule $^{(c_1, \dots, c_8)} U_{(q|U, W \cap U)}$ .

## References

- [1] Dietrich Burde, *Left-symmetric algebras, or pre-Lie algebras in geometry and physics*, arXiv:math-ph/0509016 v2 11 Nov 2005.
- [2] Nathan Jacobson, *Structure and Representations of Jordan Algebras*, Amer. Math. Soc. Colloq. Pub., Vol.39, Amer. Math. Soc., Providence, R.I., 1968.
- [3] Jean-Louis Loday, *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Enseign. Math. (2) 39 (1993), no. 3-4, 269–293.